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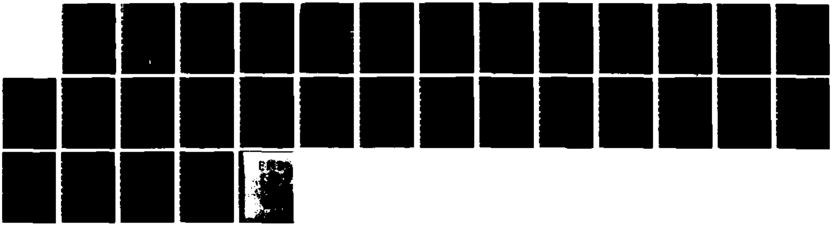

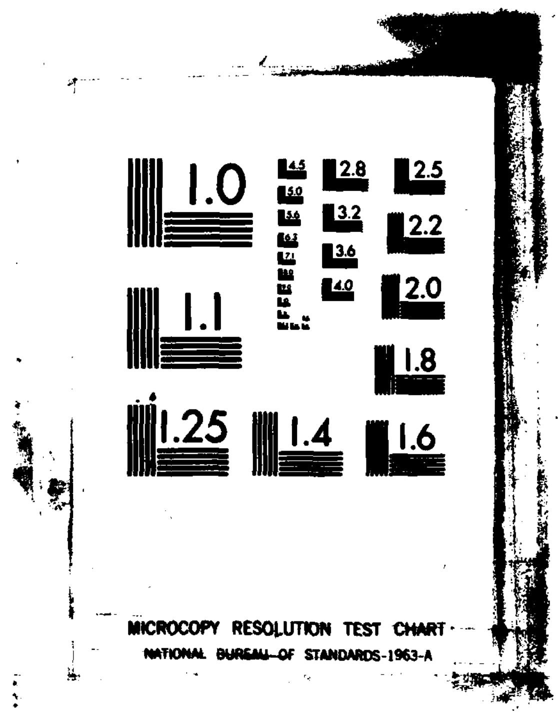
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CORRECTED DIFFUSION APPROXIMATIONS TO FIRST PASSAGE TIMES

by

Michael L. Hogan
Columbia University

TECHNICAL REPORT NO. 25
MAY 1984

PREPARED UNDER CONTRACT
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Abstract

Let X_i be i.i.d., $EX_i = 0$, $EX_i^2 = 1$, $S_n = X_1 + \ldots + X_n$, g be a reasonable function, and $T_m = \inf\left\{n : S_n > m^{1/2} g\left(\frac{T_m}{m}\right)\right\}$. The limiting distribution of $S_{T_m} - m^{1/2} g\left(\frac{T_m}{m}\right)$ is found, and used to compute heuristically higher-order correction terms to the passage probability, $P\left\{\frac{T_m}{m} < t\right\}$, the first term of which is given by the invariance principle.

Key Words: Random Walk, Diffusion Approximation, Passage Times, Tests of Power 1.

Much of the motivation and interest of the problem considered in this paper arises where the underlying object of interest is the time at which a random walk exits from a certain region. Typically this cannot be calculated exactly and some sort of approximation is necessary, which proceeds by imbedding the boundary of the region in an infinite family of similar boundaries, allowing them to move out to ∞ and calculating the distribution of the exit time asymptotically. Imbedding the boundary in a family of boundaries, by the method employed here, is equivalent to imbedding the random walk in a family of random walks, and considering the time at which they cross a fixed boundary. This can be done in such a way that the imbedded random walks tend to become degenerate along the line of drift, and thus exit the region in a small neighborhood of where the line of drift exited. This means that only the properties of the boundary at the point at which the line of drift crosses affect the distribution of the excess over the boundary and the crossing time. This seems to be a very bad property, because boundaries which are entirely different but happen to be tangent at one point appear the same to random walks which tend to cross at that point, and one fears that the asymptotic calculation has misled. The monograph of Woodroofe [1982] discusses this case.

An alternative technique, to which the above objection cannot be made, is to scale the random walk so that it converges to a Wiener process, possibly with drift. Then, the probability that the random walk crosses a boundary in a given interval of time approaches the probability of the same event for the Wiener process. The main objection that one can make to this calculation is that anything that distinguishes one distribution from another is wiped out in taking the limit. The proposed remedy is to calculate higher order correction terms to the Wiener process crossing probabilities which are typically highly distribution dependent.

Section 1 in this document takes up the question of finding the distribution of excess over the boundary for random walks normalized to converge to a Wiener process. In some cases, such as tests of power 1 which are discussed in Section 2, the connection of the excess over the boundary to the problem of finding the correction to the Wiener process crossing time is simple and direct, and in other cases, especially

that of Siegmund [1979], quantities connected with the excess over the boundary appear in ways that do not shed much light on the general problem. Section 3 gives a heuristic method for correcting the random walk probabilities, based on a reflection principle method of Siegmund and Yuh, which is the author's best idea for the connection between the excess over the boundary and the corrected diffusion approximations to the hitting times in the general case. The results of Section 1 are used here.

1. Excess Over the Boundary in Diffusion Approximations

Here is some notation that will be used in Theorem 1.1. Let $g_+(t), g_-(t), 0 \leq t \leq 1$ be continuously differentiable functions such that $g_+ > 0$, $g_- < 0$. Let S_n be a nonarithmetic random walk with $E|S_1|^3 < \infty$, $ES_1 = 0$, and $ES_1^2 = 1$. Let $T_m = \inf\{n \leq m : S_n > m^{1/2} g_+(\frac{n}{m}))$ or $S_n < m^{1/2} g_-(\frac{n}{m})\}$. Let

$$R_+(T_m) = \left(S_{T_m} - m^{1/2} g_+\left(\frac{T_m}{m}\right)\right) 1_{\{S_{T_m} > 0\}},$$

and $R(S_1)$ be a random variable with the distribution of excess over a distant boundary. That is, if $\tau_a = \inf\{n : S_n > a\}$, then $S_{\tau_a} - a \xrightarrow{L} R_+(S_1)$. The convergence in distribution is a consequence of the renewal theorem. $\mathcal{F}_a$ denotes the field generated by $\{X_i 1_{(a \geq i)}\}$, where $X_i = S_i - S_{i-1}$. $L(X)$ will denote the distribution of the random variable X, and $(t_1, t_2) \subset [0,1]$. Let $W(t)$ be a standard Wiener process, and

$$T = \inf\{t \leq 1 : W(t) > g_+(t) \text{or} W(t) < g_-(t)\}.$$

Theorem 1.1. With notation as above, and $(t_1, t_2) \subset [0,1]$,

$$P\left\{\frac{T_m}{m} \in (t_1, t_2), S_{T_m} > 0, R_+(T_m) > y\right\} \to P\{T \in (t_1, t_2), W(t) > 0\} P\{R(S_1) > y\}.$$

Furthermore, if $E|S_1|^\gamma < \infty$, $\gamma \geq 3$ then there is a random variable $Z \geq 0$ with $EZ^{\gamma-3} < \infty$, and $P\{S_{T_m} > 0, R_+(T_m) > y,\ T_m/m \leq 1\} \leq \text{const.} P\{Z > y\}$.

The facts about the hitting time itself are a standard, though not completely trivial, application of the invariance principle. Specifically

$$\frac{1}{m} \inf\left\{n : S_n > m^{1/2} g\left(\frac{n}{m}\right)\right\} = \inf_n \left\{\frac{n}{m} < 1 : \frac{S_n}{m^{1/2}} > g\left(\frac{n}{m}\right)\right\}$$
$$\xrightarrow{L} \inf\{t < 1 : W(t) > g(t)\}.$$

It is therefore taken as given that $\frac{T_m}{m} \xrightarrow{L} T$. The key to establishing Theorem 1.1 is Lemma 1.1, which implies that even as the boundary moves out to ∞ and smooths out, the process can get within k units of it, where k is some positive constant, without (with high probability) crossing. The proof is then completed by observing that starting from reasonably close to the boundary the process crosses as if it were crossing a distant constant boundary.

Lemma 1.1. The 0(1) Boundary Crossing Lemma. With notation as in Theorem 1, there is a $k > 0$ and a random variable $Z > 0$ with $EZ^{\gamma-3} < \infty$ depending on the boundary only through $\alpha = \sup_{t\in[0,1]} g'_+(t)$, such that

$$P\{R_+(T_m) > y, \frac{T_m}{m} < 1\} \leq kP\{Z > y\}.$$

Proof. The distribution of excess over the boundary can be written

$$\begin{aligned}
P\{R_+(T_m) > y, \frac{T_m}{m} \leq 1\} &= \sum_{j=0}^{m} P\left\{S_{j+1} > m^{1/2}\, g_+\left(\frac{j+1}{m}\right) + y,\ T_m > j\right\} \\
&= \sum_{j=0}^{m} \int_{z=-\infty}^{0} P\left\{S_{j+1} - S_j > m^{1/2}\left(g_+\left(\frac{j+1}{m}\right)\right) + y + z\right\} \\
&\times P\left\{S_j \in m^{1/2}\, g_+\left(\frac{j}{m}\right) - dz,\ T_m > j\right\} \\
&\leq \int P\{S_{j+1} - S_j > 1 + y + z\} \qquad\qquad (1.1) \\
&\times \sum P\left\{S_j \in m^{1/2}\, g\left(\frac{j}{m}\right) - dz,\ T_m > j\right\} \\
&\leq \sum_{n=1}^{\infty} P\{S_{j+1} - S_j > n + y\} \\
&\times \sum_{j=1}^{m} P\left\{S_j \in \left(m^{1/2}\, g_+\left(\frac{j}{m}\right) - n,\ m^{1/2}\, g_+\left(\frac{j}{m}\right) - n + 1\right),\ T_m > j\right\},
\end{aligned}$$

where the next to the last line follows from $m^{1/2}|g_+\left(\frac{j+1}{m}\right), -g_+\left(\frac{j}{m}\right)| \leq |\frac{\alpha}{m^{1/2}}| \leq 1$ for $m^{1/2}$ large. It is necessary, therefore, to bound

$$EV_n = \sum^{m} P\left\{S_j \in \left(m^{1/2}\, g_+\left(\frac{j}{m}\right) - n,\ m^{1/2}\, g_+\left(\frac{j}{m}\right) - n + 1\right),\ T_m > j\right\}.$$

This is the expected number of visits to a certain set by the space-time random walk before exiting from a larger set. Assume that the random walk has the

property that $P\{S_{n^2} > (1+\alpha)n\} > \gamma > 0\ \forall n$. Assume that $n < m^{1/2}$. Then

$$\begin{aligned} P\{V_n > k + n^2 | V_n > k\} &\le P\left\{S_{n^2} \le \alpha\, \frac{n^2}{m^{1/2}} + n\right\} \\ &\le P\{S_{n^2} \le (1+\alpha)n\} \\ &\le 1-\gamma. \end{aligned}$$

The first inequality merely bounds the probability of spending n^2 more time units in the set before exiting the larger set by the probability that in precisely n^2 more steps the random walk has exited the larger set. It is easy to see that this implies $EV_n \le cn^2$, where the c depends only on γ, which depends on the boundary through α only. Also, because $T_m < m$, $EV_n \le cn^2$ is very obvious for $n > m^{1/2}$. To remove the restriction imposed above, note that the CLT guarantees that the inequality holds for $n > n_0$ and the entire argument can be repeated on $P\{V_n > k + n_0 n^2 | V_n > k\}$ to produce a bound $EV_n \le cn^2$, where c could depend on n_0. So the bound $EV_n \le cn^2$, where c depends on the boundary only through α has been established. Substituting this bound into (1.1) gives

$$P\left\{R_+(T_m) > y,\ \frac{T_m}{m} \le 1\right\} \le c \sum_{m=1}^{\infty} n^2\, P\{S_1 > y + n\}.$$

Now if $E|S_1|^3 < \infty$, Z can be chosen to be the random variable with distribution

$$P\{Z > y\} = \frac{\sum_{n=1}^{\infty} n^2 P\{S_1 > y+n\}}{\sum_{n=1}^{\infty} n^2 P\{S_1 > n\}}.$$

If $\gamma \le 3$ the statement about the moments follows from

$$\begin{aligned} EZ^{\gamma-3} &\le \text{const.} \sum_{j=0}^{\infty} j^{\gamma-4}\, P\{Z > j\} \\ &= \text{const.} \sum_{j=0}^{\infty} j^{\gamma-4} \sum_{n=j}^{\infty} n^2\, P\{S_1 > j+n\} \\ &= \text{const.} \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} j^{\gamma-4}\, n^2\, P\{S_1 > j+n\} \\ &\le \text{const.} \sum_{n=1}^{\infty} n^{\gamma-1}\, P\{S_1 > n\} \\ &< \infty, \end{aligned}$$

since $E|S_1|^\gamma < \infty$, and where the constant is not necessarily the same from line to line.

Proof of Theorem. Let

$$T_m^k = \inf\left\{n \le m = S_n > m^{1/2} g_+\left(\frac{n}{m}\right) - k \text{ or} S_n < m^{1/2} g_-\left(\frac{n}{m}\right) + k\right\},$$

where it is assumed that the relation beteen k and m is such that $m^{1/2} g_+(\frac{n}{m}) - k > 0$, $m^{1/2} g_-(\frac{n}{m}) + k < 0$. Denote by

$$R_+(T_m^k) = \left(S_{T_m^k} - \left(m^{1/2} g_+\left(\frac{T_m}{m}\right) - k\right)\right) 1_{(S_{T_m} > 0)},$$

and similarly for $R_-(T_m^k)$. Lemma 1 applies to $R_\pm(T_m^k)$, so, $\epsilon > 0$ being given, it is possible to choose k so that $P\{R_\pm(T_m^k) > \frac{k}{2}\} < \epsilon$. It may also be assumed that k is large. Fix an interval (t_1, t_2). The first step is to show that those paths for which $T_m \in (mt_1, mt_2)$ and $S_{T_m} > 0$ are essentially those for which $T_m^k \in (mt_1, mt_2)$ and $S_{T_m^k} > 0$. The symmetric difference between the two sets is contained in

$$\begin{aligned}
&\{T_m^k \in (mt_1, mt_2),\ S_{T_m^k} > 0, T_m \in (mt_1, mt_2),\ S_{T_m} < 0\} \\
&\quad \cup \{T_m^k \in (mt_1, mt_1),\ T_m \notin (mt_1, mt_2)\} \\
&\quad \cup \{T_m \in (mt_1, mt_2),\ T_m^k \notin (mt_1, mt_2)\} \\
&\quad \cup \{T_m^k \in (mt_1, mt_2),\ S_{T_m^k} < 0,\ T_m \in (mt_1, mt_2),\ S_{T_m} > 0\}.
\end{aligned} \tag{1.2}$$

Consider the first set. Let $u = \sup\{g_-(t), t \in (t_1, t_2)\}$, $\ell = \inf\{g_+(t), t \in (t_1, t_2)\}$. By hypothesis $u < 0 < \ell$. By the strong Markov property

$$\begin{aligned}
&P\{T_m^k \in (mt_1, mt_2),\ S_{T_m^k} > 0,\ T_m \in (mt_1, mt_2),\ S_{T_m} < 0\} \\
&\qquad \le P^{\circ}\{\tau_{(-\infty, k+\alpha)} > \log m\} + P^{\circ}\{S_{\tau_{(um^{1/2}, k+\alpha)}} < u\, m^{1/2}\},
\end{aligned}$$

where $\tau_I = \inf\{t : S_t \notin I\}$, and P^x denotes the measure governing the random wlk started from the point x at time 0. This is so because within $\log m$ after T_m^k it exits when the upper boundary is within

$$k + m^{1/2}\left(g\left(\frac{T_m^k + \lambda \log m}{m}\right) - g\left(\frac{T_m^k}{m}\right)\right) \le k + \alpha,$$

of the value of the random walk at time T_m^k, and where $0 < \lambda < 1$. It is clear that the probabilities of both the first and the second events go to 0 as $m \to \infty$.

The second set in (1.2) is also easy to handle. Similar to the above

$$\begin{aligned} &P\{T_m^k \in (mt_1, mt_2), T_m \notin (mt_1, mt_2)\} \\ &\qquad \leq P\{T_m^k \in (mt_2 - m^{1/2}, mt_2)\} \\ &\qquad + P^\circ\{T_{(-\infty, k+\alpha)} > m^{1/2}\} + P^\circ\{T_{(-k-\alpha, \infty)} > m^{1/2}\}. \end{aligned}$$

It is easy to show that the exit distribution of the Wiener process through boundaries considered here is continuous. Consequently, the invariance principle shows that the first probability goes to 0, and the second and third approach 0 as above. (Strassen [1967] shows that in the one boundary case, the exit distribution actually has a continuous density. His argument presumably extends to the two-sided case, but this strong a result is not necessary here.)

The last two probabilities can be handled analogously.

Therefore, the above argument shows that regardless of k, for m sufficiently large,

$$\begin{aligned} &P(\{T_m \in (mt_1, mt_2),\ S_{T_m} > 0,\ R_+(T_m) > y\} \\ &\qquad \Delta\{T_m^k \in (mt_1, mt_2),\ S_{T_m^k} > 0, R_+(T_m) > y\}) < \epsilon/2, \end{aligned}$$

where $\epsilon \geq 0$ is arbitrary. So for k sufficiently large, Lemma 1 guarantees that

$$\begin{aligned} &P(\{T_m \in (mt_1, mt_2),\ S_{T_m} > 0,\ R_+(T_m) > y\} \\ &\qquad \Delta(A_m^k \cap R_+(T_m) > y)) < \epsilon, \end{aligned}$$

where $A_m^k = \{T_m^k \in (mt_1, mt_2),\ S_{T_m^k} > 0, R_+(T_m^k) < \frac{k}{2}\}$. The next part of the argument will show that conditionally on $\mathcal{F}_{T_m^k}$, on A_m^k, exiting the region is much like crossing a distant constant boundary. Some extra notation must, alas, be introduced to show this.

If Z_n is any random walk with $0 \leq EZ_1 < \infty$ denote by $t_a = \inf\{n : Z_n > a\}$, $R_a(Z_1) = Z_{t_a} - a$, and $R_\infty(Z_1)$ a random variable with the distibution $\mathcal{L}(R_\infty(Z_1)) = \lim_{a\to\infty} \mathcal{L}(R_a(Z))$, which limit exists by the renewal theorem. The starting point is the following identity:

$$P\{A_m^k \wedge \{R_+(T_m) > y\} = \int_{A_m^k} P\{R_+(T_m) > y | \mathcal{F}_{T_m^k}\}\}dP. \qquad (1.3)$$

Consider

$$
\begin{aligned}
P\{R_+(T_m) > y \mid \mathcal{F}_{T_m^k}\} \\
&= P\{R_+(T_m) > y, T_m - T_m^k < \log m \mid \mathcal{F}_{T_m^k}\} \qquad (1.4)\\
&\quad + P\{R_+(T_m) > y,\ T_m - T_m^k > \log m \mid \mathcal{F}_{T_m^k}\}.
\end{aligned}
$$

As in the proof of the bounds in (1.2), the second probability goes to 0 for m sufficiently large, regardless of k. To bound the first probability, note that over an interval of length $\log m$ the boundary varies little, that is

$$\sup_{j \le \log m} m^{1/2} \left| g\left(\frac{T_m^k + j}{m}\right) - g\left(\frac{T_m^k}{m}\right)\right| \le \alpha \frac{\log m}{m^{1/2}} \to 0.$$

Let us pick an $\epsilon_1 \le \alpha \frac{\log m}{m^{1/2}}$ and compare the value of T_m to

$$T_m^{\pm\epsilon_1} = \inf\left\{j > T_m^k : S_j - S_{T_m^k} > m^{1/2}\, g\left(\frac{T_m^k}{m}\right) - S_{T_m^k} \pm \epsilon_1\right\}.$$

On the set A_m^k, $m^{1/2}\, g\left(\frac{T_m^k}{m}\right) - S_{T_m^k} > k/2$. Suppose k is chosen so that $\inf_{a > k/4} P\{R_a(S_1) > 2\epsilon_1\} > 1 - \epsilon$. Then on A_m^k

$$P\{T_m^{\epsilon} > T_m^{-\epsilon} \mid \mathcal{F}_{T_m^k}\} = P^{S_{T_m^k}}\left\{R_{m^{1/2}\, g(T_m^k/m) - S_{T_m^k} - \epsilon_1}(S_1) < 2\epsilon_1 \mid \mathcal{F}_{T_m^k}\right\} \le \epsilon.$$

So since $T_m^{\epsilon_1} \ge T_m^{-\epsilon_1}$, $P\{T_m^{\epsilon} = T_m^{-\epsilon} \mid \mathcal{F}_{T_m^k}\} \ge 1 - \epsilon$ on A_m^k. Also it is clear that

$$\{T_m^{\epsilon_1} = T_m^{-\epsilon_1},\ T_m - T_m^k < \log m\} \subset \{T_m^{\epsilon_1} = T_m\},$$

and further, that the excess over the three boundaries differs by $< 2\epsilon_1$ on this set. Therefore, choosing k so that

$$\sup_{a \ge k/4} |P\{R_a > y + \epsilon_1\} - P\{R_a > y - \epsilon_1\}| < \epsilon,$$

for m large the first term on the right hand side of (1.4) can be bounded above and below as follows:

$$
\begin{aligned}
P\{R(T_m^{-\epsilon_1}) > y + 2\epsilon,\ T_m^{\epsilon_1} - T_m^k < \log m \mid \mathcal{F}_{T_m}\} - P\{T_m^{\epsilon_1} \neq T_m \mid \mathcal{F}_{T_m}\} \\
&\le P\{R(T_m) > y, T_m - T_m^k < \log m \mid \mathcal{F}_{T_m}\} \\
&\le P\{R(T_m^{\epsilon_1}) > y,\ T_m^{\epsilon_1} - T_m < \log m \mid \mathcal{F}_{T_m}\} + P\{T_m^{-\epsilon} \neq T_m \mid \mathcal{F}_{T_m}\}.
\end{aligned}
$$

Considering only the upper bound, it in turn can be bounded by

$$P\{R(T_m^{\epsilon}) > y|\mathcal{F}_{T_m}\} + P\{T_m^{\epsilon} \neq T_m|\mathcal{F}_{T_m}\}.$$

For m large enough, the second term can be made $< \frac{\epsilon}{2}$ on A_m^k. Also, k can be chosen sufficiently large so that

$$\sup_{a \geq k/4} |P\{R_a(S_1) > y\} - P\{R_\infty(S_1) > y\}| < \epsilon.$$

Then on A_m^k

$$|P\{R(T_m^{\epsilon}) > y|\mathcal{F}_{T_m}\} - P\{R_\infty(S_1) > y\}| < \epsilon.$$

Substituting these various bounds into (1.4) gives

$$\begin{aligned} P\{T_m \in (mt_1, mt_2),\ S_{T_m} > 0,\ R_+(T_m) > y\} &\leq P\{A_m^k \cap (R(T_m) > y)\} + \epsilon \\ &\leq P\{A_m^k\}P\{R_\infty > y\} + 4\epsilon \\ &\leq P\{T_m^k \in (mt_1, mt_2),\ S_{T_m^k} > 0\}P\{R_\infty > y\} + 5\epsilon \\ &\leq P\{T \in (t_1, t_2),\ W(t) > 0\}P\{R_\infty > y\} + 6\epsilon \end{aligned}$$

for m large by the invariance principle. The lower bound can be established similarly, completing the proof.

Remarks.

1) The one-boundary case follows from the two-boundary case by letting the lower boundary (say) $\to -\infty$.

2) The restriction $g_+ > 0$, $g_- < 0$ is not necessary, it can be replaced by $g_-(t) < g_+(t)$, $0 \leq t \leq 1$. It is, of course, necessary to assume that $g_-(0) < 0 < g_+(0)$, because otherwise the Wiener process crosses immediately at 0 (or the lower boundary, say, is actually an upper boundary).

3) A drift $\frac{\mu}{m^{1/2}}$ can be added to the random walk at stage m to make it converge to a Wiener process with drift, and this is equivalent to adding a linear term to the boundary. In particular, the excess over the boundary is the same. Other ways of getting a Wiener process with drift are possible, e.g. at stage m choose the member of a fixed, one parameter exponential family with expectation

$\frac{\mu}{m^{1/2}}$ to generate the random walk. The main new tool needed for this case is a uniform rate of convergence for R_a to R_∞, over members of the exponential family in a neighborhood of the distribution with drift. In an exponential family, it is not hard to show (possibly with the additional assumption of nonarithmeticity) that $R_\infty(S_1^\theta)$ is a continuous function of θ. Therefore, if for large a the quantities R_a for the different distributions are all close to their limiting value, they must all be close to the value of R_∞ for the mean zero member. It is believed that an argument of this sort can replace the part of the proof following (1.4), and that this can be made rigorous using a coupling argument, but details have not been worked out. Siegmund and Yuh [1982] consider this framework. They use a different method of establishing this uniform convergence.

4) There is a clear conjecture for what happens in the lattice case. Consider the case of a single upper boundary, denoted by g. Suppose that S_1 is arithmetic with span 1. The technique of the proof above shows that the distribution of $R_+(T_m)$ is approximately $R_\infty(S_1) - \langle m^{1/2}\, g\left(\frac{T_m}{m}\right)\rangle$, where $\langle x \rangle = x - [x]$ = fractional part of x, and where $R_\infty(S_1)$ is independent of T_m. The random variable $g\left(\frac{T_m}{m}\right)$ is converging in distribution, and, if g is not constant, the limiting random variable, $g(T)$, has a reasonably smooth density. When X is any bounded random variable with a continous density it is not hard to show that $\langle m^{1/2} X\rangle \xrightarrow{\mathcal{L}} U$, where U is a uniform random variable on $[0,1]$. Therefore the obvious conjecture is that $R_+(T_m) \xrightarrow{\mathcal{L}} R_\infty(S_1) - U$, $R_\infty(S_1)$, U independent. There is a problem with rates of convergence, or perhaps local uniformity in time of convergence that prevents this argument from being rigorous.

2: Application to a Test of Power 1.

The reader shoule recall that the main motivation for this work is the correction of the power functions of statistical tests defined by boundary crossing. It is assumed that the crossing distribution for the Wiener process is known, or very well approximable and the problem is to correct this crossing probability for discreteness or non-normality. The following is an example where information about the excess over the boundary alone suffices to obtain the first correction term of the Wiener process boundary crossing probabilities. The crossing problem below is corrected with tests known as tests of power 1, and are discussed in

Woodroofe [1982], Chapter 6.

Let P_μ be a family of probabiity measures under which the process $\{S_t;\ t \geq 0\}$ is a Wiener process with mean μt and variance t. Define a measure Q by

$$Q(A) = \int_{-\infty}^{\infty} P_\mu(A)\phi(\mu)d\mu,$$

where Q is a standard, normal distribution. At time t, $Q << P_0$ and the likelihood ratio is given by

$$L_t = \frac{dQ}{dP_0}|_t = \left(\frac{1}{1+t}\right)^{1/2} e^{-S_t^2/2(1+t)}.$$

Let $c > 1$ and define a stopping time T by

$$\begin{aligned} T &= \inf\{t : L_t > c\} \\ &= \inf\{t : |S_t| > g(t)\} \end{aligned}$$

where

$$g(t) = (2(t+1)(\log c + \log(1+t)))^{1/2}.$$

The fundamental identity of sequential analysis (a.k.a. Wald's likelihood ratio identity, see Siegmund [6] propositions 2.24 and 3.2) shows that

$$\begin{aligned} P_0\{T < \infty\} &= E_Q\left(\frac{1}{L_T};\ T < \infty\right) \\ &= \frac{1}{c}, \end{aligned}$$

since $P_\mu\{T < \infty\} = 1\ \forall \mu \neq 0$. The discrete version of this test defines a stopping time

$$T_m = \inf\{n \in Z^+;\ L_{n/m} > c\},$$

which, because of the scaling properties of the Wiener process can be written as being equal in law to

$$\inf\left\{n : |S_n| > m^{1/2}\, g\left(\frac{n}{m}\right)\right\}. \tag{2.1}$$

Theorem 2.1.

$$P_0\{T_m < \infty\} = \frac{1}{c} - \frac{1}{m^{1/2}} E_0(R_\infty(S_1)) \times E_0\left\{\frac{g(T)}{2(T+1)};\ T < \infty\right\} + o\left(\frac{1}{m^{1/2}}\right).$$

Proof. By the fundamental identity

$$\begin{aligned} P_0\{T_m < \infty\} &= E_Q\left\{\frac{1}{L_{T_m}};\ T_m < \infty\right\} \\ &= E_Q\left\{\frac{1}{L_{T_m}}\right\} \\ &= \frac{1}{c}\int_{-\infty}^{\infty} E_\mu\left\{exp - \left\{\frac{g(T_m/m)}{2(T_m/m+1)}R_\pm + \frac{R_\pm^2}{2(T_m/m+1)}\right\}\right\}\phi(\mu)d\mu \qquad (2.2) \\ &= \frac{1}{c}\int_{-\log m}^{\log m} E_\mu\left\{exp - \left\{\frac{g(T_m/m)}{2(T_m/m+1)}R_\pm + \frac{R_\pm^2}{2(T_m/m+1)}\right\}\right\}\phi(\mu)d\mu \\ &\quad + o\left(\frac{1}{m^{1/2}}\right) \end{aligned}$$

where $R_\pm$ is the excess above or below the boundary. In this scale one expects $R_\pm = 0\left(\frac{1}{m^{1/2}}\right)$. Each of the summands in the brackets in (2.2) is non-negative, and $R_\pm$ is always a positive quantity. A one term Taylor expansion gives that

$$\begin{aligned} &P_0\{T_m < \infty\} = \\ &\frac{1}{c}\left\{1 - \frac{1}{m^{1/2}}\int_{-\log m}^{\log m} E_\mu\left\{\left(\frac{g(T_m/m)}{2(T_m/m+1)}(m^{1/2}R_\pm) + \frac{(m^{1/2}R_\pm)^2}{m^{1/2}(T_m/m+1)}\right)f_m(\cdot)\right\}\right\} \\ &\quad \phi(\mu)d\mu + o\left(\frac{1}{m^{1/2}}\right), \end{aligned} \qquad (2.3)$$

where $f_m(\cdot)$ is a random quantity satisfying $|f_m(\cdot)| < 1$ and $f_m(\cdot) \to 1$ as $m \to \infty$. It is clear how the integrand should behave. Theorem 1.1 gives the asymptotic distribution of $m^{1/2}R_\pm$, and the fact that it is asymptotically independent of $\frac{T_m}{m}$, $\frac{g(t)}{1+t}$ is and continuous bounded, so the first summand is converging in distribution, and the second is converging to 0. The only fly in the ointment is that Theorem 1.1 does not give quite enough information to claim uniform integrability for $|m^{1/2}R_\pm|$ or $|m^{1/2}R_\pm|^2$, which also must be uniform over μ. This must be taken care of by a special argument.

Under P_μ, T_m can be written

$$T_m = \inf\left\{n : \left|\frac{S_n}{m^{1/2}} - \mu \cdot \frac{n}{m}\right| > g\left(\frac{n}{m}\right)\right\}$$
$$\stackrel{\mathcal{L}}{=} \inf\left\{n : \left|S_n - \frac{\mu}{m^{1/2}} \cdot n\right| > m^{1/2} g\left(\frac{n}{m}\right)\right\},$$

where S_n is a $N(0,1)$ random walk. That is, T_m is the sort of stopping time considered in Theorem 1, with drift $\frac{\mu}{m^{1/2}}$. Similarly, $R_\pm$ can be written as being equal in law to

$$\frac{1}{m^{1/2}} \left| g\left(\frac{T_m}{m}\right) - S_{T_m}\right| = \frac{1}{m^{1/2}} |R_m|.$$

The technique is very much like the proof of Theorem 1.1. Suppose one wants to establish bounds on the distribution function for excess above the upper boundary. This distribution can be written

$$P.\{R_m > y\} = \int_z P.\{S_1 > y + z\}$$
$$\times \sum_{m=0}^{\infty} P.\left\{S_n \in m^{1/2} g\left(\frac{n+1}{m}\right) - dz,\ T_m > n\right\}$$

and consequently

$$P.\{R_m > j\} \leq \sum_{m=0}^{\infty} P.\{S_1 > j + n\}$$
$$\times \sum_{\ell=0}^{\infty} P.\left\{S_\ell \in m^{1/2} g\left(\frac{\ell+1}{m^{1/2}}\right) - (n, n+1)\right\},$$

as in Theorem 1.1. If the bound of const. n^5 is obtained on the second sum, the R_m has been stochastically bounded by a distribution with moments of all orders, as in Theorem 1.1. Therefore, we seek to control

$$E\sum^{T_m} 1_{(j,j+1)}\left(S_n + m^{1/2} g\left(\frac{n+1}{m}\right)\right)$$

in the case of the lower boundary or

$$E\sum^{T_m} 1_{(j,j+1)}\left(m^{1/2} g\left(\frac{n+1}{m}\right) - S_n\right) \tag{2.4}$$

in the case of the upper. There are naturally two cases, the drift 0 case which works equally well for small drifts, $\frac{k}{m^{1/2}}$, with $|k| < k_0$, and the case of large drifts. In the zero drift case the upper and lower boundary are the same, but in the positive drift case this is not so. It will, however, be clear in the positive drift case that the proof of the zero drift case works equally well with respect to boundedness above the upper boundary, so only excess below the lower boundary need be considered. The case of a sizeable negative mean follows by symmetry.

Case 1. Zero Drift. The argument for Theorem 1.1 provides a bound on (2.3) up to some time $\xi_0 m$ and also for visits within $k_0 m^{1/2}$ of the boundary. Therefore assume $j > k_0 m^{1/2}$. Note that

$$g'(x) = \frac{\log c + 1 + \log(1+x)}{2((1+x)(\log c + \log(1+x)))^{1/2}} \sim \frac{1}{2}\left(\frac{\log(1+x)}{x}\right)^{1/2}.$$

Suppose that $n > j^5$. Then

$$\begin{aligned} &P\{\text{exiting within } j^5 \text{ more steps} \mid \text{within } j \text{ of boundary at time } n\} \\ &\qquad \geq P\left\{S_{j^2} \geq m^{1/2}\left(g\left(\frac{n+j^5}{m}\right) - g\left(\frac{n}{m}\right)\right) + j\right\} \\ &\qquad \geq P\left\{S_{j^5} \geq \frac{1}{m^{1/2}}\, g'\left(\frac{j^4}{m}\right)\cdot j^2 + j\right\}. \end{aligned} \tag{2.5}$$

But it is easy to see that

$$\frac{1}{m^{1/2}}\, g'\left(\frac{j^5}{m}\right)\cdot j^2 \leq j,$$

and hence by scale properties of the normal distribution the quantity in (2.4) is bounded away from 0. This supplies a bound of const. (j^5) on (2.3), and the proof of the claim is finished as in Theorem 1.1. Note, as explained above, that if the random walk had been [illegible] upwards this would have bounded the excess uniformly above provided that the mean drift is bounded above, which is only to say the increments are stochastically bounded.

Case 2. Positive Drift. Let $\mu = \frac{k}{m^{1/2}}$, $k > 1$. Consider j in two classes,

$j < \frac{1}{\mu}$ and $j \geq \frac{1}{\mu}$. For $j \geq \frac{1}{\mu}$

$$\begin{aligned} P_\mu\{S_{n+j^2} > -m^{1/2}\, g\left(\frac{n+j^2}{m}\right) + 2j+1 | S_n \in g\left(\frac{n+j^2}{m}\right) + (j, j+1)\} \\ \geq P_\mu\{S_{j^2} - j^2\mu \geq 2j+1-j^2\mu\} \\ \geq P_\mu\{S_{j^2} - j^2\mu > 2j+1\} > \epsilon > 0 \,\forall j. \end{aligned}$$

And by Lemma 2 below

$$P\{\text{ever being within } j \text{ of boundary } | \text{farther than } 2j \text{ from boundary}\} \leq \epsilon^{-2}.$$

This provides a bound on (2.3) in the manner of Theorem 1.1. On the other hand, for $j \leq \frac{m^{1/2}}{k} - \frac{1}{\mu}$, since g' is bounded

$$\begin{aligned} P\{S_{n+j^2} < -m^{1/2}\, g\left(\frac{n+j^2}{m}\right) | S_n + m^{1/2}\, g\left(\frac{n}{m}\right) + (j, j+1)\} \\ \geq P\{S_{j^2} - j^2\mu < j - \frac{j^2}{m^{1/2}} - j^2\mu\}. \end{aligned}$$

But $\frac{j}{m^{1/2}} \leq \frac{j\mu}{k} < 1$, the left hand side of the inequality is a $N(0, j^2)$ and the right hand side is (uniformly) within its standard deviation. This gives a bound on (2.3) of const. (j^2) as above, and finishes the proof of the claims.

Now it has been shown that the quantities represented by $m^{1/2}R_\pm$ in (2.3) are uniformly integrable for all values of the parameter. To finish the proof of the theorem it is a simple matter to take the limit in (2.3) and using the asymptotic independence of $m^{1/2}R_\pm$ and T_m, and the form of the limiting distribution given in Theorem 1.1, to write

$$\begin{aligned} P_0\{T_m < \infty\} &= \frac{1}{c}\left\{1 + \frac{E_0(R_\infty(S_1))}{m^{1/2}} \int_{-\infty}^{\infty} E_\mu \frac{g(T)}{2(T+1)} \phi(\mu) d\mu + o\left(\frac{1}{m^{1/2}}\right)\right\} \\ &= \frac{1}{c} + \frac{E_0(R_\infty(S_1))}{m^{1/2}} \frac{1}{c} E_Q \frac{g(T)}{2(T+1)} + o\left(\frac{1}{m^{1/2}}\right) \\ &= \frac{1}{c} + \frac{E_0(R_\infty(S_1)}{m^{1/2}} E_0 \left\{\frac{g(T)}{2(T+1)};\, T < \infty\right\}, \end{aligned}$$

where the definitions of Q and of the stopping time have been used. This finishes the proof of the claim.

Lemma 2.1. Let $\mu > 0$, $x > 0$, $x_0 > 0$, and $\tau_x = inf\{n : S_n > x\}$. Then

$$P_{-\mu}\{\tau_{x_0/\mu} < \infty\} \le e^{-2x_0}.$$

Proof.

$$\begin{aligned} P_{-\mu}\{\tau_{x_0/\mu} < \infty\} &= E_\mu\{\exp(-2\mu(S_{\tau_{x_0/\mu}}))\} \\ &= e^{-2x_0} E_\mu\{\exp(-2\mu(S_{\tau_{x_0/\mu}} \frac{x_0}{\mu}))\}, \end{aligned}$$

and $S_{\tau_{x_0/\mu}} - \frac{x_0}{\mu} \ge 0$.

The form of the approximation in Theorem 2.1 illustrates the main theoretical and practical drawback: it depends globally on the Wiener process crossing distribution. Practically this means that to use this method it would be necessary to compute the crossing distribution for the Wiener process, which is usually impossible to write down explicitly. Theoretically, related questions such as $P_0\left\{\frac{T_m}{m} < 1\right\}$ necessitate having to approximate $P_\theta\left\{\frac{T_m}{m} < 1\right\}$ which does not involve simply the excess over the boundary. A method for approximating $P_0\left\{\frac{T_m}{m} < 1\right\}$ for certain random walks and boundaries is the subject of Section 3.

3. Corrected Diffusion Approximations to First Passage Times.

First, some notation will be introduced and summarized. It will be assumed that the probability space will accomodate a random walk, S_n, with $ES_1 = 0$, $ES_1^2 = 1$, $ES_1^3 = 0$, and a standard Wiener process $W(t)$. T's will be used for random walk crossing times and τ's for those of the Wiener process. τ_g will denote different things in the one-sided and two-sided case; the reader may consider himself fairly warned. It will generally be clear what boundary the random walk is crossing, and T_m will denote that crossing time for $\frac{S_n}{m^{1/2}}$. If $t_a = \inf\{n : S_n > a\}$, then it is known from renewal theory that $S_{t_a} - a$ converges in distribution and is uniformly integrable. Set $\beta = \lim_{a\to\infty} E\, S_{\tau_a} - a$. The results are as follows:

(1) Let $g < 0$ be continuously differentiable on $[0,1]$, $t < 1$,

$$T_m = \inf\left\{n : S_n > m^{1/2}\, g\left(\frac{n}{m}\right)\right\},$$

and $\tau_f = \inf\{t : W(t) > f(t)\}$ for any function f. Then

$$P\left\{\frac{T_m}{m} < t\right\} = P\{\tau_{g+\beta/m^{1/2}} < t\} + o\left(\frac{1}{m^{1/2}}\right)$$

and

$$P\left\{\frac{T_m}{m} < t,\ \frac{S_{mt}}{m^{1/2}} < g(t) - x\right\} = P\{T_{g+\beta/m^{1/2}} < t,\ W(t) < g(t) - x\} + o\left(\frac{1}{m^{1/2}}\right).$$

(2) Assume that X_1 is symmetric. Let $g > 0$ be continuously differentiable on $[0,1]$, $0 \leq t_0 < t_1 < 1$, $T_m = \inf\{n \geq mt_0 : |S_n| > m^{1/2} g(\frac{n}{m})\}$, and $\tau_h = \inf\{t \geq t_0 : |W(t)| > h(t)\}$ for any function h. Then

$$\begin{aligned} P\left\{\frac{T_m}{m} < t_1\right\} &= 2\left(1 - \Phi\left(\frac{g(t_0)}{t_0^{1/2}}\right)\right) \\ &\quad + P\{t_0 < \tau_{g+\beta/m^{1/2}} < t_1\} + o\left(\frac{1}{m^{1/2}}\right) \end{aligned}$$

and

$$\begin{aligned} &P\left\{t_0 < \frac{T_m}{m} < t_1,\ |S_{mt_1}| < m^{1/2}(g(t_1) - x)\right\} \\ &= P\{t_0 < \tau_{g+\beta/m^{1/2}} < t_1,\ |W(t_1)| < g(t_1) - x\} + o\left(\frac{1}{m^{1/2}}\right). \end{aligned}$$

Notice that because of the assumed symmetry of the random walk, to consider nonsymmetric terminal sets in (2) is no increase in generality.

(3) Let P_θ be a measure such that $P_\theta\{X_1 \in A\} = P_0\{X_1 - \theta \in A\}$, and $P = P_0$. Assume that X_1 is symmetric under P. Let P^ξ be a measure under which the Wiener process has drift ξ. Then

$$\begin{aligned} P_{\xi/m^{1/2}}\left\{\frac{T_m}{m} < t\right\} &= P^\xi\{|W(t_0)| > g(t_0)\} \\ &\quad + P^\xi\{t_0 < \tau_{g+\beta/m^{1/2}} < t_1\} + o\left(\frac{1}{m^{1/2}}\right). \end{aligned}$$

Remarks.

(1) The results essentially say that the corrected probabilities for the random walks are the probabilities that the Wiener process crosses the boundary pushed

out by an amount equal to the expected excess over the boundary. This has intuitive appeal, and it agrees with the linear cases computed by Siegmund and Yuh [1982].

(2) There is no increase in generality in allowing starting times other than 0 for the one-sided boundary problem. This can be done by conditioning on the position of the random walk at the appropriate time. The two-sided problem requires symmetric boundaries and so conditional on the position of the random walk at a fixed time the boundaries appear unsymmetric. Evidently the symmetry of the random walk makes this effect cancel out on the average.

(3) There are many variants of these probabilities that it would be nice to do, but about which this method says nothing. Foremost is the case of nonzero skewness. In that case, this method will derive integral equations for the correction term as below, but no solutions can be exhibited, except in the case of a constant boundary, where the results agree with Siegmund and Yuh's. In the two-sided case, it would be nice to be able to calculate

$$P\left\{\frac{T_m}{m} < 1,\ S_{T_m} < 0\right\},$$

as well as to be able to handle nonsymmetric boundaries. The author opines that these two problems are linked, and that there is an "interaction" term between the upper and lower boundaries which makes both problems difficult, but which cancels out if sufficient symmetry is present.

(4) In, for example, a normal translation family, if a test is defined by a one-sided boundary, crossing probabilities for contiguous alternatives, which are those whose expected value is $\frac{\text{const.}}{m^{1/2}}$, follow directly from these results, since the drift can be added to the boundary. However, in the two-sided case the result is once again messed up by the introduction of asymmetry into the problem. The two-sided case is the more interesting one for applications.

(5) S. V. Nagaev [1970] established a Berry-Essen type bound of $\frac{\text{const.}}{m^{1/2}}$ for this problem, showing that, in the above notation,

$$\left|P\left\{\frac{T_m}{m} < t\right\} - P\{\tau < t\}\right| \leq \frac{L(\|g'\|_\infty + 1)(E|X_1|^3)^2}{m^{1/2}},$$

where L is an absolute constant, by using basically a Fourier inversion method. That analysis is very difficult. A review of the state of boundary crossing problems as of 1972 can be found in Borovkov [1972].

Heuristic Proof of the Result: There are two major points that make the proof heuristic. One will be mentioned when it arises but the other is at the heart of the proof. It is assumed that

$$P\left\{\frac{T_m}{m} < t\right\} = P\{\tau < t\} + \frac{C(t)}{m^{1/2}} + o\left(\frac{1}{m^{1/2}}\right),$$

where $C(t)$ is a smooth function. An integral equation is then derived for $C(t)$, and a solution to the integral equation is exhibited.

Consider now the one-sided boundary problem, so that

$$T_m = \inf\left\{n : S_n > m^{1/2} g\left(\frac{n}{m}\right)\right\}.$$

The starting point for this analysis, as for that of Siegmund and Yuh, is a simple decomposition of all paths into those which are above the boundary at time t and those which are not.

$$\begin{aligned} P\left\{\frac{T_m}{m} < t\right\} &= P\left\{\frac{S_{mt}}{m^{1/2}} > g(t)\right\} + P\left\{\frac{T_m}{m} < t, \frac{S_{mt}}{m^{1/2}} < g(t)\right\} \\ &= P\left\{\frac{S_{mt}}{m^{1/2}} > g(t)\right\} + \sum_{j=1}^{mt-1} P\left\{S_{mt} - S_j < m^{1/2}\left(g(t) - g\left(\frac{j}{m}\right)\right) - R,\ T_m = j\right\} \end{aligned}$$

where R is the excess over the boundary, $S_{T_m} = m^{1/2}\, g\left(\frac{T_m}{m}\right) + R$. The above is

$$P\left\{\frac{S_{mt}}{m^{1/2}} > g(t)\right\} + \sum_{j=1}^{mt-1} P\left\{S_{mt} - S_j < m^{1/2}\left(g(t) - g\left(\frac{j}{m}\right)\right) - R_j\right\} P\{T_m = j\}.$$

Now R_j is a random variable that has the distribution of R conditioned on $\{T_m =$

$j\}$, and is independent of $S_{mt}-S_j$. Now by an Edgeworth expansion this becomes

$$1-\Phi\left(\frac{g(t)}{t^{1/2}}\right)+\sum^{mt-1} E\Phi\left(\left(\frac{m}{mt-j}\right)^{1/2}\left(g(t)-g\left(\frac{j}{m}\right)\right)-\frac{R_j}{(m-j)^{1/2}}\right)$$
$$\times P\left\{\frac{T_m}{m}=\frac{j}{m}\right\}+o\left(\frac{1}{m^{1/2}}\right)$$
$$\doteq 1-\Phi\left(\frac{g(t)}{t^{1/2}}\right)+\sum^{mt-1}\Phi\left(\frac{g(t)-g(j/m)}{\sqrt{t-j/m}}\right)P\left\{\frac{T_m}{m}=\frac{j}{m}\right\}$$
$$-\sum^{mt-1}\phi\left(\frac{g(t)-g(j/m)}{\sqrt{t-j/m}}\right)\frac{ER_j}{(m-j)^{1/2}}P\left\{\frac{T_m}{m}=\frac{j}{m}\right\}.$$

Here another set of heuristic assumptions are introduced. First, the last sum represents a quantity of the form $\frac{1}{m^{1/2}}E\{f(T_m)R;\ T_m<mt\}$, where the quantity $f(T_m)$ has a singularity like $\frac{1}{(t-T_m/m)^{1/2}}$ at the endpoint. It is assumed first of all that the distribution of T_m is such that this quantity has a uniformly bounded $1+\epsilon^{th}$ moment for some $\epsilon\geq 0$. Note that this claim is true for the Wiener process. Siegmund and Yuh also ran into this problem as they note in [1982], and recently Siegmund has circumvented it without bounding higher moments. Nevertheless, let that stand as one assumption. Second, under further assumptions on the random walk, the asymptotic joint distribution of R and T_m is known from Section 1. It will be assumed that there is a corresponding local limit theorem which says that $ER_j\doteq ER=\beta$. Combining these observations with the invariance principle, and neglecting terms that are $o\left(\frac{1}{m^{1/2}}\right)$, the following equation for the hitting probabilities results:

$$P\left\{\frac{T_m}{m}<t\right\}=1-\Phi\left(\frac{g(t)}{t^{1/2}}\right)+\sum^{mt-1}\Phi\left(\frac{g(t)-g(j/m)}{\sqrt{t-j/m}}\right)P\left\{\frac{T_m}{m}=\frac{j}{m}\right\}$$
$$-\frac{\beta}{m^{1/2}}E\left(\frac{\phi(g(t)-g(\tau)/(t-\tau)^{1/2})}{(t-\tau)^{1/2}};\ \tau=\tau_g<t\right).$$

One further simplification is possible. One sees, upon conditioning on $\mathcal{F}_{\tau_g}$, that the last expectation $\times dx$ represents $P\{W(t)\in g(t)+dx,\ \tau_g<t\}$. However

$$\frac{\phi(g(t)/t^{1/2})}{t^{1/2}}dx=P\{W(t)\in g(t)+dx,\ \tau_g<t\}+P\{W(t)\in g(t)+dx,\ \tau_g\geq t\}$$
$$=P\{W(t)\in g(t)+dx,\ \tau_g<t\}.$$

Therefore

$$P\left\{\frac{T_m}{m} < t\right\} = 1 - \Phi\left(\frac{g(t)}{t^{1/2}}\right) + \sum^{mt-1} \Phi\left(\frac{g(t) - g(j/m)}{\sqrt{t - j/m}}\right) \times P\left\{\frac{T_m}{m} = \frac{j}{m}\right\} - \frac{\beta}{m^{1/2}} \frac{\phi(g(t)/t^{1/2})}{t^{1/2}}. \tag{3.1}$$

The last term will be called the $0\left(\frac{1}{m^{1/2}}\right)$ part of the equation, and the rest the $0(1)$ part of the equation. This is to be regarded as an equation for $P\left\{\frac{T_m}{m} < t\right\}$ or $P\left\{\frac{T_m}{m} = j\right\}$. The strategy is to produce a function that satisfies the $0(1)$ part of the equation except for terms which are $o\left(\frac{1}{m^{1/2}}\right)$, and to represent the solution as a perturbation of this function. The candidate is $f\left(\frac{j}{m}\right) = P\left\{\tau_g \in \left(\frac{j-1}{m}, \frac{j}{m}\right)\right\}$. To see that this works, consider those paths for which $\tau_g \in \left(\frac{j-1}{m}, \frac{j}{m}\right)$. The increment $W\left(\frac{j}{m}\right) - W(\tau_g)$ is small $\left(E\left(W\left(\frac{j}{m}\right) - W(\tau_g)\right)^2 = 0\left(\frac{1}{m}\right)\right)$, and symmetrically distributed. Therefore, the distribution H, defined by

$$W\left(\frac{j}{m}\right) 1_{\tau_g \in \left(\frac{j-1}{m}, \frac{j}{m}\right)} = \left(g\left(\frac{j}{m}\right) + H\right) 1_{\tau_g \in \left(\frac{j-1}{m}, \frac{j}{m}\right)}$$

also satisfies $EH^2 = 0\left(\frac{1}{m}\right)$, and $EH = 0\left(\frac{1}{m}\right)$. Most importantly, conditioning on τ_g shows that $f\left(\frac{j}{m}\right)$ exactly satisfies

$$F(t) = 1 - \Phi\left(\frac{g(t)}{t^{1/2}}\right) + \sum^{mt-1} E\left(\Phi\left(\frac{g(t) - g(j/m) - H}{\sqrt{t - j/m}}\right)\right) f\left(\frac{j}{m}\right)$$

where $F(t) = \sum^{mt} f\left(\frac{j}{m}\right)$. A simple Taylor expansion now shows that $F(t)$ satisfies the $0(1)$ part of the equation except for terms that are $o\left(\frac{1}{m^{1/2}}\right)$.

Now suppose that the solution to (3.1) can be written as $F(t) + \frac{\beta F_1(t)}{m^{1/2}}$ where $F_1(t) = \sum^{mt} f_1\left(\frac{j}{m}\right)$. Necessarily, $F_1(0) = 0$. Then an equation for $F_1(t)$ can be derived by substituting this in (3.1) and using the fact that $F(t)$ satisfies the $0(1)$

part of the equation. Explicitly

$$F(t)+\frac{\beta F_1(t)}{m^{1/2}}=1-\Phi\left(\frac{g(t)}{t^{1/2}}\right)+\sum^{mt-1}\Phi\left(\frac{g(t)-g(j/m)}{\sqrt{t-j/m}}\right)$$
$$\times\left(f\left(\frac{j}{m}\right)+\frac{\beta}{m^{1/2}}f_1\left(\frac{j}{m}\right)\right)-\frac{\beta}{m^{1/2}}\frac{\phi(g(t)/t^{1/2})}{t^{1/2}}+o\left(\frac{1}{m^{1/2}}\right)$$
$$F_1(t)=\sum^{mt-1}\Phi\left(\frac{g(t)-g(j/m)}{\sqrt{t-j/m}}\right)f_1\left(\frac{j}{m}\right)-\frac{\phi(g(t)/t^{1/2})}{t^{1/2}}+o(1).$$

If F_1 is smooth, $f_1\left(\frac{j}{m}\right)\doteq\frac{F_1'(\frac{j}{m})}{m}$, and neglecting terms that are $0(1)$ the equation for $F_1(t)$ is

$$F_1(t)=\int_0^t\Phi\left(\frac{g(t)-g(s)}{(t-s)^{1/2}}\right)F_1(ds)-\frac{\phi(g(t)/t^{1/2})}{t^{1/2}}. \tag{3.2}$$

As an aside, note that integration by parts converts (3.2) into a Fredholm equation of the second kind. The associated homogeneous equation is

$$F_1(t)=\int_0^t\Phi\left(\frac{g(t)-g(s)}{(t-s)^{1/2}}\right)F_1(ds)$$

(which is homogeneous because $F_1(0)=0$). In order for the nonhomogeneous equation to have a unique solution it is necessary and sufficient that the homogeneous equation should have no nontrivial solution. It is likely that this is the case, however the solution that will be produced has sufficient intuitive appeal to make questions about its uniqueness almost moot.

Our candidate for a solution is

$$\frac{\partial}{\partial h}P\{\tau_{g+h}<t\}\,|_{h=0}\,.$$

Substituting this in (3.2) shows that it must satisfy

$$\frac{\partial}{\partial h}P\{\tau_{g+h}<t\}\,|_{h=0}=\int_0^t\Phi\left(\frac{g(t)-g(s)}{(t-s)^{1/2}}\right)\frac{\partial}{\partial s}\frac{\partial}{\partial h}P\{\tau_{g+h}<s\}\,|_{h=0}\,ds-\frac{\phi(g(t)/t^{1/2}}{t^{1/2}}.$$

There should be no problem interchanging orders of differentiation, or in taking $\frac{\partial}{\partial h}$ outside the integral, and so doing yields for the right hand side above

$$\frac{\partial}{\partial h}\int_0^t\Phi\left(\frac{g(t)-g(s)}{(t-s)^{1/2}}\right)P\{\tau_{g+h}\in ds\}-\frac{\phi(g(t)/t^{1/2})}{t^{1/2}}. \tag{3.3}$$

The integral has a clear probabilistic interpretation. Consider the Wiener process and condition on $\mathcal{F}_{\tau_{g+h}}$, then clearly the integral above represents

$$P\{\tau_{g+h} < t,\ W(t) < g(t) + h\} = P\{\tau_{g+h} < t\} - P\{W(t) > g(t) + h\}.$$

And so the expression in (3.3) is

$$\begin{aligned}&\frac{\partial}{\partial h} P\{\tau_{g+h} < t\}\ |_{h=0} - \frac{\partial}{\partial h} P\{W(t) > g(t) + h\}\ |_{h=0}\\ &\qquad - \frac{\phi(g(t)/t^{1/2})}{t^{1/2}} = \frac{\partial}{\partial h} P\{\tau_{g+h} < t\}\ |_{h=0},\end{aligned}$$

which is to say that $\frac{\partial}{\partial h} P\{\tau_{g+h} < t\}\ |_{h=0}$ is a solution.

What this means is that

$$\begin{aligned}P\left\{\frac{T_m}{m} < t\right\} &= P\{\tau_g < t\} + \frac{\beta}{m^{1/2}} \frac{\partial}{\partial h} P\{\tau_{g+h} < t\}\ |_{h=0} + o\left(\frac{1}{m^{1/2}}\right)\\ &= P\{\tau_{g+\beta/m^{1/2}} < t\} + o\left(\frac{1}{m^{1/2}}\right)\end{aligned}$$

as the expression in the first line is the Taylor series expansion of that in the second.

The second statement of 1 follows directly from the first part:

$$\begin{aligned}&P\left\{\frac{T_m}{m} < t,\ S_{mt} < g(t) - x\right\} = \sum^{mt} \Phi\left(\frac{g(t) - x - g(j/m) - R_j}{(mt - j)^{1/2}}\right) P\left\{\frac{T_m}{m} = \frac{j}{m}\right\}\\ &= \int_0^t \Phi\left(\frac{g(t) - g(s) - x}{(t - s)^{1/2}}\right) P\{\tau_{g+\beta/m^{1/2}} \in ds\}\\ &- \frac{\beta}{m^{1/2}} \int_0^t \frac{\phi(g(t) - g(s) - x)/(t - s)^{1/2})}{(t - s)^{1/2}} P\{\tau_{g+\beta/m^{1/2}} \in ds\} + o\left(\frac{1}{m^{1/2}}\right)\\ &= P\left\{\tau_{g+\beta/m^{1/2}} < t,\ W(t) < g(t) - x + \frac{\beta}{m^{1/2}}\right\}\\ &- P\{\tau_{g+\beta/m^{1/2}} < t,\ W(t) \in \left(g(t) - x,\ g(t) - x + \frac{\beta}{m^{1/2}}\right) + o\left(\frac{1}{m^{1/2}}\right)\\ &= P\{\tau_{g+\beta/m^{1/2}} < t,\ W(t) < g(t) - x\} + o\left(\frac{1}{m^{1/2}}\right).\end{aligned}$$

Now consider the case of a two-sided symmetric boundary and symmetric random walk. Here $0 < t_1 < t < 1$, $T_m = \inf\{n > mt_1 : |S_n| > m^{1/2}\, g(t)\}$, $\tau_g =$

$\inf\{t > t_1 : |W(t)| > m^{1/2} g(t)\}$. The analysis that leads to the integral equation is very similar to that of the one-sided case, and the reason for the restriction to symmetric situations does not show up in the derivation, except insofar as it leads to a different integral equation to which the author can present no solution, and therefore it will be omitted. The equation that $P\left\{\frac{T_m}{m} < t\right\}$ should satisfy is

$$P\left\{\frac{T_m}{m} < t\right\} = 2(1 - \Phi(g(t_1)) + 2(1 - \Phi(g(t_2)))$$
$$+ \sum_{mt_1}^{mt_2} \left(\Phi\left(\frac{g(t) + g(j/m)}{\sqrt{t - j/m}}\right) - \Phi\left(\frac{g(j/m) - g(t)}{\sqrt{t - j/m}}\right)\right) P\left\{\frac{T_m}{m} = \frac{j}{m}\right\}$$
$$+ \frac{\beta}{m^{1/2}} E\left(\frac{\phi(g(t) + g(\tau)/(t-\tau)^{1/2}) - \phi(g(t) - g(\tau)/(t-\tau)^{1/2})}{(t-\tau)^{1/2}}; t_1 < \tau < t\right).$$

As before, some analysis is possible on the last expression

$$E\left(\frac{\phi(g(t) + g(\tau)/(t-\tau)^{1/2}}{(t-\tau)^{1/2}}; t_1 < \tau < t\right) dx$$
$$= 2P\{W(t) \in g(t) + dx,\ t_1 < \tau < t,\ W(t) < 0\}$$

where the factor of 2 comes in because each boundary contributes symmetrically, and

$$E\left(\frac{\phi(g(t) - g(\tau)/(t-\tau)^{1/2})}{(t-\tau)^{1/2}}\ t_1 < \tau < t\right) dx$$
$$= 2P\{W(t) \in g(t) + dx,\ t_1 < \tau < t,\ W(\tau) < 0\}.$$

Therefore, the sum of these two terms is $2P\{W(t) \in g(t) + dx,\ t_1 < \tau < t\}$, and so the difference can be represented as

$$4P\{W(t) \in g(t) + dx,\ W(\tau) < 0,\ t_1 < \tau < t\} - 2P\{W(t) \in g(t) + dx,\ t_1 < \tau < t\}.$$

Call this $H(t)dx$.

As above, the Wiener process solution, $F_0(t)$, can be shown to satisfy the $0(1)$ part of the equation up to terms that are $o\left(\frac{1}{m^{1/2}}\right)$, and so, if a solution of the form $F_0(t) + \frac{\beta}{m^{1/2}} F_1(t)$ is posited, it is easily seen that $F_1(t)$ should satisfy

$$F_1(t) = \int_{t_1}^{t} \Phi\left(\frac{g(t) + g(s)}{(t-s)^{1/2}}\right) - \Phi\left(\frac{g(t) - g(s)}{(t-s)^{1/2}}\right) F_1(ds) + H(t).$$

The proposed solution is $\frac{\partial}{\partial h} P\{t_1 < \tau_{g+h} < t,\ |W(t_1)| < g(t_1)\}$. Substituting this in the integral gives

$$\frac{\partial}{\partial h}\int_{t_1}^{t}\left(\Phi\left(\frac{g(t)+g(s)}{(t-s)^{1/2}}\right)-\Phi\left(\frac{g(t)-g(s)}{(t-s)^{1/2}}\right)\right)P\{\tau_{g+h}\in ds,\ |W(t_1)|< g(t_1)\},$$

which is the $\frac{\partial}{\partial h}$ of

$$\begin{aligned}
&P\{t_1 < \tau_{g+h} < t,\ W(\tau_{g+h}) > 0,\ -g(t)+h < W(t) < g(t)+h,\ |W(t_1)|< g(t_1)\}\\
&\quad + P\{t_1 < \tau_{g+h} < t,\ W(\tau_{g+h}) < 0,\\
&\quad -g(t)-h < W(t) < g(t)-h,\ |W(t_1)|< g(t_1)\}\\
&= P\{t_1 < \tau_{g+h} < t,\ W(\tau_{g+h} > 0,\ -g(t)-h < W(t) < g(t)+h,\ |W(t_1)|< g(t_1)\}\\
&\quad + P\{\tau_{g+h} < t,\ W(\tau_{g+h}) < 0,\ -g(t)-h < W(t) < g(t)+h,\ |W(t_1)|< g(t_1)\}\\
&\quad - P\{t_1 < \tau_{g+h} < t,\ W(\tau_{g+h}) > 0,\\
&\quad -g(t)-h < W(t) < -g(t)+h,\ |W(t_1)|< g(t_1)\}\\
&\quad - P\{t_1 < \tau_{g+h} < t,\ W(\tau_{g+h}) < 0,\\
&\quad g(t)-h < W(t) < g(t)+h, |W(t_1)|< g(t_1)\}\\
&= P\{t_1 < \tau_{g+h} < t,\ |W(t_1)|< g(t_1)\}\\
&\quad - P\{|W(t)|> g(t)+h,\ t_1 < \tau_{g+h},\ |W(t_1)|< g(t_1)\}\\
&\quad - 2P\{t_1 < \tau_{g+h} < t,\ W(\tau_{g+h}) < 0,\\
&\quad -g(t)-h < W(t) < -g(t)+h,\ |W(t_1)|< g(t_1)\}\\
&= P\{t_1 < \tau_{g+h} < t,\ |W(t_1)|< g(t_1)\} - P\{|W(t)|> g(t)+h,\ t_1 < \tau_g\}\\
&\quad - 2P\{t_1 < \tau_{g+h} < t,\ W(\tau_{g+h}) < 0,\\
&\quad -g(t)-h < W(t) < g(t)+h, |W(t_1)|< g(t_1)\}.
\end{aligned}$$

The $\frac{\partial}{\partial h}$ of this expression is easily seen to be

$$\begin{aligned}
&\frac{\partial}{\partial h}P\{t_1 < \tau_{g+h} < t,\ |W(t_1)|< g(t_1)\}\ |_{h=0} + 2P\{W(t)\in g(t)+dx,\ t_1 < \tau_g\}/dx\\
&\quad - 4P\{t_1 < \tau_g < t,\ W(\tau_g) < 0,\ W(t)\in -g(t)+dx\}/dx,
\end{aligned}$$

and comparing this with the expression for $H(t)$ given above shows that

$$\frac{\partial}{\partial h}P\{\tau_{g+h} < t,\ |W(t_1)|< g(t)\}$$

is a solution. This means that

$$\begin{aligned}
P\left\{\frac{T_m}{m} < t\right\} &= P\left\{|S_{mt_1}|> m^{1/2}\, g(t_1)\right\}\\
&\quad + P\{t_1 < \tau_{g+\beta/m^{1/2}} < t,\ |W(t_1)|< g(t_1)\} + o\left(\frac{1}{m^{1/2}}\right).
\end{aligned}$$

The second statement of 2 follows as did the second statement of 1.

The proof of the third result is a little bit different. Once again it can be argued that if

$$P\left\{\frac{T_m}{m} < t\right\} = F(t) + \frac{F_1(t)}{m^{1/2}} + o\left(\frac{1}{m^{1/2}}\right),$$

then $\frac{F_1(t)}{\beta}$ satisfies an integral equation. The exact form of this equation is not of any interest except that it must be noted that it does not depend in any way (other than symmetry) on the particular random walk. Now, the equation itself constitutes an invariance principle for the correction term, and if the correction term can be found for any member of the family by any method, it is necessarily the correction term for other symmetric random walks. A likelihood-ratio method will now be used to find the correction term for normal random walk.

Suppose, therefore, that the random walk is Gaussian. We will use below, in order of appearance, Wald's likelihood ratio identity (Siegmund [6], propositions 2.24 and 3.2) on the random walk, Theorem 1.1, result 2 of this section, and the likelihood ratio identity on the resulting diffusion limits.

$$\begin{aligned}
P_{\xi/m^{1/2}}\left\{\frac{T_m}{m} < t\right\} &= E_0\left\{e^{\xi/m^{1/2} S_{T_m} - \xi^2/2\, T_m/m};\ \frac{T_m}{m} < t\right\} \\
&= E_0\left\{e^{\xi g(T_m/m) + \xi\frac{R_m}{m^{1/2}} - \xi^2/2\, T_m/m};\ \frac{T_m}{m} < t\right\} \\
&= E_0\left\{e^{\xi g(T_m/m) - \xi^2/2\, T_m/m}\left(1 + \xi\frac{R_m}{m^{1/2}} + O\left(\frac{R_m}{m^{1/2}}\right)^2\right);\ \frac{T_m}{m} < t\right\} \\
&= E_0\left\{e^{\xi g(T_m/m) - \xi^2\, T_m/2m}\left(1 + \frac{\xi\beta}{m^{1/2}}\right);\ \frac{T_m}{m} < t\right\} + o\left(\frac{1}{m^{1/2}}\right) \\
&= E_0\left\{e^{\xi((g+\beta/m^{1/2})(T_m/m)) - \xi^2/2\, T_m/m};\ \frac{T_m}{m} < t\right\} + o\left(\frac{1}{m^{1/2}}\right) \\
&= \int_0^t e^{\xi((g+\beta/m^{1/2})(s)) - (\xi^2/2)\, s} P_0\left\{\frac{T_m}{m} \in ds\right\} + o\left(\frac{1}{m^{1/2}}\right) \\
&= \int_0^t e^{\xi(g+\beta/m^{1/2}(s)) - \xi^2/2\, s} P_0\left\{\tau_{g+\beta/m^{1/2}} \in ds\right\} + o\left(\frac{1}{m^{1/2}}\right) \\
&= E_0\left\{e^{\xi W(\tau_{g+\beta/m^{1/2}}) - \xi^2/2\, \tau_{\beta/m^{1/2}}};\ \tau_{g+\beta/m^{1/2}} < t\right\} + o\left(\frac{1}{m^{1/2}}\right) \\
&= P^{\xi}\left\{\tau_{g+\beta/m^{1/2}} < t\right\} + o\left(\frac{1}{m^{1/2}}\right).
\end{aligned}$$

This implies the stated result.

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